CONTRIBUTIONS TO THE THEORY OF RAMANUJAN'S FUNCTION $\tau(n)$ AND SIMILAR ARITHMETICAL FUNCTIONS

II. THE ORDER OF THE FOURIER COEFFICIENTS OF INTEGRAL MODULAR FORMS

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1. Suppose that
$$H(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau/N} \tag{1·1}$$
is an integral modular form of dimensions $-\kappa$, where $\kappa > 0$, and *Stufe* N, which vanishes at all the rational cusps of the fundamental region, and which is absolutely convergent for $y = \Im\tau > 0$. Then
$$H(\tau) = H_1(\tau) = (c\tau+d)^{-\kappa} H_i\left(\frac{a\tau+b}{c\tau+d}\right), \tag{1·2}$$
where a, b, c, d are integers such that $ad-bc = 1$.

Let Γ be the non-homogeneous congruence group of transformations for which
$$H(\tau) = (c\tau+d)^{-\kappa} H\left(\frac{a\tau+b}{c\tau+d}\right),$$
and let μ be the index of Γ. Then the number of different functions $H_i(\tau)$ is G/μ, where G is the total number of different non-homogeneous transformations congruent modulo N. It is known that*
$$G = \frac{N^3}{2\rho} \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \tag{1·3}$$
where p is a prime factor of N, and
$$\rho = \tfrac{1}{2} \quad (N = 1, 2), \qquad \rho = 1 \quad (N > 2).$$

Theorem 1. $|a_1|^2 + |a_2|^2 + \dots + |a_n|^2 = \alpha n^\kappa + O(n^{\kappa-\frac{2}{5}})$ *as n tends to infinity, where*
$$\alpha = 12 \frac{(4\pi)^{\kappa-1}}{\mu N^\kappa \Gamma(\kappa+1)} \iint y^{\kappa-2} |H(\tau)|^2 \, dx dy, \tag{1·4}$$
the double integral being taken over any fundamental region of Γ.

* Cf. for example, F. Klein and R. Fricke, *Elliptische Modulfunktionen*, 1 (Leipzig, 1890), 395–7.

THEOREM 2. $$a_n = O(n^{\frac{1}{2}\kappa-\frac{1}{5}}).$$

Theorem 2 is an immediate corollary from Theorem 1 and is a slight improvement on the previous result obtained independently by Salié* and Davenport†, viz.

$$a_n = O(n^{\frac{1}{2}\kappa-\frac{1}{6}+\epsilon}),$$

for any $\epsilon > 0$. The method which is used here does not depend on the estimation of Kloosterman sums.

2. *Remarks*. **A**. Theorems 1 and 2 give, when applied to the modular form

$$\Delta(\tau) = e^{2\pi i\tau} \prod_{\nu=1}^{\infty} (1-e^{2\pi i\nu\tau})^{24} = \sum_{n=1}^{\infty} \tau(n)\, e^{2\pi i n\tau},$$

the result that $$\tau^2(1)+\tau^2(2)+\ldots+\tau^2(n) \sim \alpha n^{12},$$

and $$\tau(n) = O(n^{\frac{29}{5}}),$$

where $\tau(n)$ is Ramanujan's function, and α is given by (1·4) on substituting $\kappa = 12$, $\mu = 1$.

B. The results of the present paper hold, not only for ordinary modular forms, but also for any modular relative invariant which satisfies

$$H_1(\tau) = \epsilon(c\tau+d)^{-\kappa} H_i\left(\frac{a\tau+b}{c\tau+d}\right) \quad (ad-bc = 1),$$

where ϵ is a constant of unit modulus depending only on a, b, c, d.

C. Let $\Gamma(N)$ be the non-homogeneous principal congruence group‡, i.e. the group of transformations

$$\tau = \frac{a\tau_1+b}{c\tau_1+d},$$

for which $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N.$$

Then the constant α in (1·4) may be written in the form

$$\alpha = 12\frac{(4\pi)^{\kappa-1}}{GN^{\kappa}\Gamma(\kappa+1)}\iint_D y^{\kappa-2}\,|\,H(\tau)\,|^2\,dx\,dy, \tag{2·1}$$

where G is given by (1·3), and D is any fundamental region of $\Gamma(N)$. For D contains G/μ fundamental regions of Γ. This is the form in which we actually obtain α in § 5.

* H. Salié, "Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen", *Math. Z.* 36 (1931), 263–78.

† H. Davenport, "On certain exponential sums", *J. reine angew. Math.* 169 (1932), 158–76.

‡ *Hauptkongruenzgruppe*.

3. We need the following lemmas.

LEMMA 1. *If D is a fundamental region of the congruence group $\Gamma(N)$ which contains the point at infinity, then*

$$\iint_D y^{-\gamma} \mid H(\tau) \mid^2 dx\,dy$$

is absolutely convergent for any γ.

This is an immediate consequence of the fact that $H(\tau)$ vanishes at all the rational vertices of D.

LEMMA 2. *If $\tau = x+iy$, where $y>0$, then*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi w}{\beta\gamma y} \mid m\beta\tau+n\gamma+\lambda\tau+\mu\mid^2\right\}$$
$$= \frac{1}{w} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi}{\beta\gamma y w} \mid m\beta\tau+n\gamma\mid^2 + \frac{2\pi i}{\beta\gamma}(m\beta\mu-n\gamma\lambda)\right\},$$

where β, γ, w are positive, and λ, μ are real, both series being absolutely convergent.

This lemma is easily obtained by applying the Poisson summation formula twice to the first double series. By elementary methods of approximation, we have

LEMMA 3. *In the notation of Lemma 2,*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi w}{\beta\gamma y} \mid m\beta\tau+n\mid^2\right\}$$
$$\leqslant 1+A_1(1+w^{-1}+y^{-\frac{1}{2}}w^{-\frac{1}{2}}+y^{\frac{1}{2}}w^{-\frac{1}{2}})\,(e^{-\pi\beta y w/2\gamma}+e^{-\pi w/2\beta\gamma y}),$$

where A_1 is a constant depending on β, γ but not on w, τ.

LEMMA 4. *If $\chi(n)$ is a primitive non-principal character modulo k, and if w, β are positive,*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \chi(n) \exp\left\{-\frac{\pi w}{\beta k y} \mid m\beta\tau+n\mid^2\right\}$$
$$= \epsilon(\chi)\frac{k^{\frac{1}{2}}}{w} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{\chi}(m) \exp\left\{-\frac{\pi}{\beta k y w} \mid m\beta\tau+nk\mid^2\right\},$$

where $\epsilon(\chi)$ is a constant of unit modulus depending only on χ.

For putting $n = n_1 k+\mu$, where $0 \leqslant \mu < k$, we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \chi(n) \exp\left\{-\frac{\pi w}{\beta k y} \mid m\beta\tau+n\mid^2\right\}$$
$$= \sum_{\mu=0}^{k-1} \chi(\mu) \sum_{m=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \exp\left\{-\frac{\pi w}{\beta k y} \mid m\beta\tau+n_1 k+\mu\mid^2\right\}$$
$$= \frac{1}{w} \sum_{\mu=0}^{k-1} \chi(\mu) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi}{\beta k y w} \mid m\beta\tau+nk\mid^2 + 2\pi i m\frac{\mu}{k}\right\},$$

by Lemma 2. And since $\chi(n)$ is a primitive non-principal character*,

$$\sum_{\mu=0}^{k-1} \chi(\mu)\, e^{2\pi i m\mu/k} = \epsilon(\chi)\, k^{\frac{1}{2}} \bar{\chi}(m),$$

and this completes the proof of the lemma.

4. *The case* $N = 1$. We treat this case separately since the analysis is much simpler.

4·1. Let $f(s)$ be the function defined by the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}, \tag{4·1·1}$$

where the numbers a_n are the coefficients in (1·1). We prove first the following theorem.

THEOREM 3. *The function* $f(s)$ *defined by* (4·1·1) *has the properties:*

(i) *The series* (4·1·1) *is absolutely convergent for* $\sigma = \Re s > \kappa$.

(ii) $f(s)$ *may be continued as a meromorphic function over the whole plane.*

(iii) $f(s)$ *has a simple pole of residue* $\kappa\alpha$ *at* $s = \kappa$.

(iv) $f(s)$ *satisfies the functional equation*

$$\phi(s) = \phi(2\kappa - 1 - s), \tag{4·1·2}$$

where $$\phi(s) = (2\pi)^{-2s}\, \Gamma(s)\, \Gamma(s-\kappa+1)\, \zeta(2s-2\kappa+2) f(s).$$

(v) $\phi(s)$ *is regular over the whole plane except for simple poles at the points* $s = \kappa$, *and* $s = \kappa - 1$.

By Parseval's theorem,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(x+iy)|^2\, dx = \sum_{n=1}^{\infty} |a_n|^2 e^{-4\pi n y} \quad (y > 0).$$

Now $$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u}\, du = (4\pi n)^s \int_0^{\infty} y^{s-1} e^{-4\pi n y}\, dy$$

for $\sigma > 0$. Therefore

$$\begin{aligned}(4\pi)^{-s}\, \Gamma(s) f(s) &= \sum_{n=1}^{\infty} |a_n|^2 \int_0^{\infty} y^{s-1} e^{-4\pi n y}\, dy \\ &= \int_0^{\infty} y^{s-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(\tau)|^2\, dx\, dy = \iint_S y^{s-1} |H(\tau)|^2\, dx\, dy, \end{aligned} \tag{4·1·3}$$

say, where S is the strip $y \geqslant 0$, $|x| \leqslant \frac{1}{2}$. The inversion of the order of integration is justified, if $\sigma > \kappa$, since the integrals are dominated by their values for real s, and since $H(\tau) = O(y^{-\frac{1}{2}\kappa})$ uniformly† as $y \to +0$. Hence (4·1·3) holds for $\sigma > \kappa$. The first part of Theorem 3 follows from this.

* Cf. E. Landau, *Primzahlen*, 1 (Leipzig, 1909), 483–92.

† This may be proved in several ways; cf., for example, E. Hecke, "Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I", *Math. Ann.* 114 (1937), 1–28 (Satz 5).

Denote by D the fundamental region

$$|x| \leqslant \tfrac{1}{2}, \quad x^2+y^2 \geqslant 1.$$

The transformation

$$\tau = T(\tau_1) = \frac{a\tau_1+b}{c\tau_1+d}, \quad (ad-bc = 1),$$

will be called an *S-transformation* if it maps points τ_1 of D on to a region D_T lying in the strip S. Since $-a$, $-b$, $-c$, $-d$ give the same transformation, we suppose that

$$c \geqslant 0, \text{ and } d > 0 \text{ if } c = 0. \tag{4·1·4}$$

Then, by (4·1·3),

$$(4\pi)^{-s}\Gamma(s)f(s) = \sum_{D_T \subset S} \iint_{D_T} y^{s-1} |H(\tau)|^2 \, dx\, dy,$$

for $\sigma > \kappa$. But

$$H(\tau) = (c\tau_1+d)^\kappa H(\tau_1),$$

$$y = \frac{y_1}{|c\tau_1+d|^2},$$

$$dx\,dy = \left|\frac{d\tau}{d\tau_1}\right|^2 dx_1\,dy_1,$$

$$\left|\frac{d\tau}{d\tau_1}\right| = \left|\frac{a}{c\tau_1+d} - c\frac{a\tau_1+b}{(c\tau_1+d)^2}\right| = \frac{1}{|c\tau_1+d|^2}.$$

Hence

$$(4\pi)^{-s}\Gamma(s)f(s) = \sum_{D_T \subset S} \iint_D \frac{y^{s-1}}{|c\tau+d|^{2s-2\kappa+2}} |H(\tau)|^2 \, dx\, dy$$

$$= \iint_D y^{s-1} |H(\tau)|^2 F(s,\tau)\, dx\, dy, \tag{4·1·5}$$

say, where

$$F(s,\tau) = \sum_{D_T \subset S} \frac{1}{|c\tau+d|^{2s-2\kappa+2}} \quad (\sigma > \kappa). \tag{4·1·6}$$

We now have to determine what the subclass of S-transformations is. Suppose that

$$\tau = T(\tau_1) = \frac{a\tau_1+b}{c\tau_1+d}$$

is an S-transformation. Then, since the point $\tau_1 = \infty$ of D corresponds to a vertex of D_T,

$$\left|\frac{a}{c}\right| \leqslant \tfrac{1}{2},$$

and there can be equality only if $c = 2$, $a = \pm 1$, by (4·1·4) since $ad-bc = 1$.

If we are given any two integers c, d satisfying (4·1·4), and

$$(c, d) = 1,$$

there exists exactly one S-transformation $T(\tau_1)$ with these values of c, d.

For (i), suppose that $c \neq 2$. Then there is exactly one solution in a, b of the equation $ad - bc = 1$ for which

$$\left|\frac{a}{c}\right| < \tfrac{1}{2},$$

and, since one vertex of the triangle D_T given by this solution is inside S, and not on either of the lines $|x| = \frac{1}{2}$, the whole of D_T must lie in S.

If (ii) $c = 2$, then $a = \pm 1$. Suppose that one of these solutions gives a triangle D_T in S touching $x = -\frac{1}{2}$, for example. Then the transformation

$$T(\tau_1) + 1$$

has the same c, d and maps D on to a triangle outside S touching $x = \frac{1}{2}$, and therefore corresponds to the other solution. Hence if one solution produces an S-transformation the other does not, and *vice versa*.

From this we obtain

$$F(s,\tau) = \sum_{\substack{m=0 \\ (m,n)=1}}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{|m\tau + n|^{2s-2\kappa+2}} \quad (\sigma > \kappa),$$

where the dash denotes that $n = 1$ if $m = 0$. Multiplying each side by

$$2\zeta(2s - 2\kappa + 2),$$

we have $$\xi(s) = 2\zeta(2s - 2\kappa + 2)\, F(s,\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{|m\tau + n|^{2s-2\kappa+2}}, \tag{4·1·7}$$

where now the dash denotes that the meaningless term where $m = n = 0$ is to be excluded. Put

$$K(w) = K(w,\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \exp\left\{-\frac{\pi w}{y}|m\tau + n|^2\right\}$$

for $w > 0$, and for points τ of D. By Lemma 2,

$$1 + K(w) = \frac{1}{w}\left\{1 + K\left(\frac{1}{w}\right)\right\}. \tag{4·1·8}$$

Now

$$\left(\frac{y}{\pi}\right)^{s-\kappa+1} \Gamma(s - \kappa + 1)\, |m\tau + n|^{-2s+2\kappa-2} = \int_0^\infty w^{s-\kappa} \exp\left\{-\frac{\pi w}{y}|m\tau + n|^2\right\} dw.$$

Hence, by (4·1·7) and (4·1·8),

$$\begin{aligned}\left(\frac{y}{\pi}\right)^{s-\kappa+1} \Gamma(s - \kappa + 1)\,\xi(s) &= \int_0^\infty w^{s-\kappa} K(w)\, dw \quad (\sigma > \kappa)\\ &= \int_1^\infty w^{s-\kappa} K(w)\, dw + \int_0^1 w^{s-\kappa}\{w^{-1}K(w^{-1}) + w^{-1} - 1\}\, dw\\ &= \int_1^\infty K(w)\,(w^{s-\kappa} + w^{\kappa-s-1})\, dw + \frac{1}{(s-\kappa)(s-\kappa+1)}. \end{aligned} \tag{4·1·9}$$

The first term on the right-hand side of (4·1·9) is a regular function of s over the whole plane, for all τ of D, by Lemma 3. Hence $\xi(s)$ is a regular function of s over the whole plane except for a simple pole at $s = \kappa$, and it is clear that $\xi(s)$ satisfies the functional equation

$$\left(\frac{y}{\pi}\right)^{s-\kappa+1} \Gamma(s-\kappa+1)\,\xi(s) = \left(\frac{y}{\pi}\right)^{\kappa-s} \Gamma(\kappa-s)\,\xi(2\kappa-1-s). \qquad (4\cdot1\cdot10)$$

From (4·1·5), (4·1·6), (4·1·7) and (4·1·9), we have, for $\sigma > \kappa$,

$$2(4\pi)^{-s}\Gamma(s)\,\Gamma(s-\kappa+1)\,\zeta(2s-2\kappa+2)f(s) = \iint_D y^{s-1}\,|\,H(\tau)\,|^2\,\Gamma(s-\kappa+1)\,\xi(s)\,dx\,dy$$

$$= \pi^{s-\kappa+1}\iint_D y^{\kappa-2}\,|\,H(\tau)\,|^2 \left\{\Gamma(s-\kappa+1)\,\xi(s)\left(\frac{y}{\pi}\right)^{s-\kappa+1}\right\} dx\,dy$$

$$= \frac{\pi^{s-\kappa+1}}{(s-\kappa)\,(s-\kappa+1)}\iint_D y^{\kappa-2}\,|\,H(\tau)\,|^2\,dx\,dy$$

$$+\pi^{s-\kappa+1}\iint_D y^{\kappa-2}\,|\,H(\tau)\,|^2\int_1^\infty K(w)\,(w^{s-\kappa}+w^{\kappa-s-1})\,dx\,dy\,dw. \qquad (4\cdot1\cdot11)$$

By Lemmas 1 and 3, this last term is absolutely convergent for all s, and hence is a regular function of s over the whole plane. Therefore the left-hand side of (4·1·11) is regular except for simple poles at $s = \kappa$, $s = \kappa-1$; and, by (4·1·10), it is easily seen that $f(s)$ satisfies the functional equation

$$\phi(s) = \phi(2\kappa-1-s),$$

where
$$\phi(s) = (2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-\kappa+1)\,\zeta(2s-2\kappa+2)f(s).$$

Thus (4·1·11) defines $f(s)$ as a meromorphic function with a simple pole at $s = \kappa$ of residue

$$\kappa\alpha = 12\,\frac{(4\pi)^{\kappa-1}}{\Gamma(\kappa)}\iint_D y^{\kappa-2}\,|\,H(\tau)\,|^2\,dx\,dy.$$

This agrees with (1·4) since the integral is invariant over any fundamental region. The other parts of Theorem 3 follow since $\phi(s)$ is regular except for poles at $s = \kappa$, $s = \kappa-1$. In the strip

$$\kappa-1 < \sigma < \kappa-\tfrac{1}{2},$$

$f(s)$ may have poles corresponding to the complex zeros of $\zeta(2s-2\kappa+2)$.

4·2. *Proof of Theorems* 1 *and* 2 *for the case* $N = 1$.

Let
$$f(s)\,\zeta(2s-2\kappa+2) = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}.$$

It is a known result*, or may be deduced from Theorem 3†, that

$$\sum_{n\leqslant x} |a_n|^2 = O(x^\kappa). \tag{4·2·1}$$

And since

$$b(n) = \sum_{d^2|n} |a_m|^2 d^{2\kappa-2},$$

where $m = n/d^2$, it follows that

$$\sum_{n\leqslant x} b(n) = O(x^\kappa). \tag{4·2·2}$$

We now apply a general theorem of Landau to the results obtained in Theorem 3. In the notation of his paper‡, put

$$Z(s) = Z_0(s) = f(s+\kappa-1)\,\zeta(2s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+\kappa-1}},$$

$$c_n = b(n)\,n^{1-\kappa}, \quad \beta = 1.$$

Also, we have, by (4·1·2),

$$\Gamma(s)\,\Gamma(s+\kappa-1)\,Z(s) = \Gamma(1-s)\,\Gamma(\kappa-s)\sum_{n=1}^{\infty} e_n \lambda_n^s,$$

for $\sigma < 0$, and

$$\sum_{\lambda_n\leqslant x} |e_n|\,\lambda_n = O\Big(\sum_{n\leqslant x} b(n)\,n^{1-\kappa}\Big) = O(x),$$

by (4·2·2). It is also clear from (4·1·11) that

$$Z(s) = O(e^{\gamma|t|}),$$

uniformly in any finite strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, as $|t|$ tends to infinity.

Hence $Z(s)$ satisfies the conditions I, II, ..., VII, VIII′ and IX of Landau's theorem, with

$$\eta = \kappa+1-(\kappa-1) = 2, \quad A = 0, \quad P = 1,$$

$$g = 0, \quad \kappa' = \beta\frac{2\eta-1}{2\eta+1} = \frac{3}{5},$$

$$R(x) = \kappa\alpha\zeta(2)\,x = \tfrac{1}{6}\pi^2\kappa\alpha x.$$

Therefore we have

$$\sum_{n\leqslant x} c_n = \sum_{n\leqslant x} b(n)\,n^{1-\kappa} = \tfrac{1}{6}\pi^2\kappa\alpha x + O(x^{\frac{3}{5}}). \tag{4·2·3}$$

And from this it is easy to deduce that

$$\sum_{n\leqslant x} b(n) = \tfrac{1}{6}\pi^2\alpha x^\kappa + O(x^{\kappa-\frac{2}{5}}).$$

Now

$$|a_n|^2 = \sum_{d^2|n} b\left(\frac{n}{d^2}\right)\mu(d)\,d^{2\kappa-2}.$$

* Cf. for example, E. Hecke, *loc. cit.*, Satz 7.

† By the Wiener-Ikehara theorem we can deduce at once from Theorem 3 that

$$\sum_{n\leqslant x} |a_n|^2 \sim \alpha x^\kappa, \quad \sum_{n\leqslant x} b_n \sim \tfrac{1}{6}\pi^2\alpha x^\kappa.$$

See S. Bochner, "Ein Satz von Landau und Ikehara", *Math. Z.* 37 (1933), 1–9.

‡ E. Landau, "Über die Anzahl der Gitterpunkte in gewissen Bereichen. II", *Nachr. Ges. Wiss. Göttingen* (1915), pp. 209–43.

Hence
$$\sum_{n \leqslant x} |a_n|^2 = \sum_{n \leqslant x} \sum_{d^2 | n} b\left(\frac{n}{d^2}\right) \mu(d)\, d^{2\kappa-2}$$
$$= \sum_{d \leqslant \sqrt{x}} \mu(d)\, d^{2\kappa-2} \sum_{\lambda \leqslant x/d^2} b(\lambda)$$
$$= \sum_{d \leqslant \sqrt{x}} \mu(d) \{\tfrac{1}{6}\pi^2 \alpha x^\kappa d^{-2} + O(x^{\kappa-\frac{2}{5}} d^{-\frac{6}{5}})\}$$
$$= \alpha x^\kappa + O(x^{\kappa-\frac{1}{2}}) + O(x^{\kappa-\frac{2}{5}})$$
$$= \alpha x^\kappa + O(x^{\kappa-\frac{2}{5}}).$$

This completes the proof of Theorem 1, and therefore of Theorem 2, for the case $N = 1$.

5. *The case* $N > 1$.

5·1. Let V be the transformation
$$\tau = V(\tau_1) = \frac{\alpha\tau_1 + \beta}{\gamma\tau_1 + \delta} \quad (\alpha\delta - \beta\gamma = 1),$$
and let
$$H(\tau) = H_1(\tau) = (c\tau + d)^{-\kappa} H_V\left(\frac{a\tau + b}{c\tau + d}\right), \tag{5·1·1}$$
for
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bmod N.$$
Put
$$H_V(\tau) = \sum_{n=1}^{\infty} a_{\gamma,\delta}(n)\, e^{2\pi i n\tau/N}$$
for some choice* of α, β, and let†
$$f_{\gamma,\delta}(s) = \sum_{n=1}^{\infty} \frac{|a_{\gamma,\delta}(n)|^2}{n^s}. \tag{5·1·2}$$
Then, if $f(s)$ is defined as in (4·1·1), $f(s) = f_{0,1}(s)$, and $a_n = a_{0,1}(n)$.

If
$$g(s, \chi) = \sum_{n=1}^{N-1} \chi(n) f_{0,n}(s), \tag{5·1·3}$$
where χ is a character modulo N, then
$$f(s) = \frac{1}{\phi(N)} \sum_{\chi} g(s, \chi). \tag{5·1·4}$$

Let χ_0 be the principal character modulo N, and let $X(n)$ be the primitive character modulo N_1 which is associated with any $\chi(n)$, where $N = N_1 N_2$. $\bar{\chi}(n)$ denotes the character conjugate to $\chi(n)$.

* $|a_{\gamma,\delta}(n)|$ is not dependent on α, β.

† When $(m, n) > 1$, $(m, n, N) = 1$, we denote by $f_{m\,n}(s)$ the function $f_{\gamma,\delta}(s)$, where $\gamma \equiv m$, $\delta \equiv n \pmod N$, $(\gamma, \delta) = 1$.

THEOREM 4. *The function $f_{\gamma,\delta}(s)$ defined by* (5·1·2) *has the following properties:*

(i) *The series* (5·1·2) *is absolutely convergent for $\sigma > \kappa$.*

(ii) *$f_{\gamma,\delta}(s)$ may be continued as a meromorphic function over the whole plane.*

(iii) *$f_{\gamma,\delta}(s)$ has a simple pole of residue $\kappa\alpha$ at $s = \kappa$.*

(iv) *$g(s,\chi)$ satisfies the functional equation*

$$\phi(s,\chi) = \left(\frac{2\pi}{N}\right)^{-2s} \Gamma(s)\,\Gamma(s-\kappa+1)\,L(2s-2\kappa+2,\chi)\,g(s,\chi)$$

$$= \epsilon(X)\left(\frac{2\pi}{N}\right)^{-2(2\kappa-1-s)} \Gamma(2\kappa-1-s)\,\Gamma(\kappa-s)\,L(2\kappa-2s,\bar{X}) \sum_{d\mid N_2} \prod_{p\mid d}\left(1-\frac{\bar{X}(p)}{p^{2\kappa-2s}}\right)$$

$$\times \prod_{p\mid N_2/d}\left(1-\frac{X(p)}{p^{2s-2\kappa+1}}\right)\frac{1}{\sqrt{N_1}}\left(\frac{d}{N_2}\right)^{2\kappa-2s} \sum_{\substack{m=1\\(md,n,N)=1}}^{N/d} \sum_{n=1}^{N} \bar{X}(m) f_{md,n}(2\kappa-1-s), \tag{5·1·5}$$

where $\epsilon(X)$ is defined in Lemma 4. When $\chi = \chi_0$, $X(n) = \epsilon(X) = N_1 = 1$, for all n.

(v) *If $\chi \neq \chi_0$, $\phi(s,\chi)$ is an integral function of s over the whole plane.*

(vi) *$\phi(s,\chi_0)$ is regular except for simple poles at $s = \kappa - 1$, $s = \kappa$.*

(vii) $$|\,g(s+\kappa-1)\,L(2s,\chi)\,| = O(e^{\gamma|t|}),$$

uniformly in any finite strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, as $|t| \to \infty$.

As in § 4·1 we have

$$N\sum_{n=1}^{\infty} |\,a_{\gamma,\delta}(n)\,|^2 e^{-4\pi n y/N} = \int_{-\frac{1}{2}N}^{\frac{1}{2}N} |\,H_V(x+iy)\,|^2\,dx,$$

and we obtain in the same manner

$$N^{s+1}(4\pi)^{-s}\,\Gamma(s) f_{\gamma,\delta}(s) = \iint_S y^{s-1}\,|\,H_V(\tau)\,|^2\,dx\,dy \quad (\sigma > \kappa), \tag{5·1·6}$$

where now S is the strip $y \geqslant 0$, $|\,x\,| \leqslant \frac{1}{2}N$.

Let D be the fundamental region of the congruence group $\Gamma(N)$, which lies in S and contains the point at infinity, and let it be transformed into the region D_T by the transformation

$$\tau = T(\tau_1) = \frac{a\tau_1+b}{c\tau_1+d} \quad (ad-bc=1),$$

where $$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix} \bmod N.$$

The point $\tau_1 = \infty$ becomes the vertex $\tau = a/c$. Now consider any such region D_T for which this point is inside S or on its right-hand boundary; i.e.

$$-\frac{N}{2} < \frac{a}{c} \leqslant \frac{N}{2}.$$

Let D'_T and D''_T be the parts of D_T inside and outside S respectively. If D''_T exists, let E''_T be the part of S congruent to D''_T by one of the transformations $\tau \pm N$.

Given any two integers c, d satisfying

$$(c, d) = 1, \quad c \equiv \gamma \pmod{N}, \quad d \equiv \delta \pmod{N},$$

there is exactly one transformation $T(\tau_1)$ with these values of c, d which transforms D into a region D_T of the kind just considered. For exactly one of the N solutions in a, b of

$$ad - bc = 1, \quad \left|\frac{a}{c}\right| < \frac{N}{2},$$

satisfies $$a \equiv \alpha \pmod{N}, \quad b \equiv \beta \pmod{N}.$$

By (5·1·6)

$$N^{s+1}(4\pi)^{-s}\,\Gamma(s) f_{\gamma,\delta}(s) = \sum_T \iint_{E_T} y^{s-1} \,|\, H_V(\tau) \,|^2 \,dx\,dy,$$

where $E_T = D'_T + E''_T$, since the strip S can be completely covered by regions of the type E_T, without overlapping. Applying the transformations $T(\tau_1)$, $T(\tau_1) \mp N$ to the integrals

$$\iint_{D'_T} y^{s-1} \,|\, H_V(\tau) \,|^2 \,dx\,dy, \quad \iint_{E''_T} y^{s-1} \,|\, H_V(\tau) \,|^2 \,dx\,dy,$$

we have, as in §4·1,

$$N^{s+1}(4\pi)^{-s}\,\Gamma(s) f_{\gamma,\delta}(s) = \rho \iint_D y^{s-1} \,|\, H(\tau) \,|^2 \,\lambda_{\gamma,\delta}(s, \tau)\,dx\,dy, \tag{5·1·7}$$

for $\sigma > \kappa$, by (5·1·1), where

$$\lambda_{\gamma,\delta}(s, \tau) = \sum_{\substack{m=-\infty \\ m \equiv \gamma,\, n \equiv \delta \pmod{N} \\ (m,n)=1}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{|\, m\tau + n \,|^{2s-2\kappa+2}}, \tag{5·1·8}$$

and ρ is 1 except when $N = 2$, in which case it is $\frac{1}{2}$. The difference in the two cases is due to the fact that, when $N = 2$, $-a$, $-b$, $-c$, $-d$ give the same transformation, and

$$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bmod 2.$$

From (5·1·3), (5·1·7) and (5·1·8), we have

$$N^{s+1}(4\pi)^{-s}\Gamma(s)\,g(s,\chi) = \rho\iint_D y^{s-1}\,|\,H(\tau)\,|^2\,G(s,\tau,\chi)\,dx\,dy, \qquad (5\cdot1\cdot9)$$

where
$$G(s,\tau,\chi) = \sum_{\substack{m=-\infty\\ m\equiv 0 \,(\mathrm{mod}\, N)\\ (m,n)=1}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\chi(n)}{|\,m\tau+n\,|^{2s-2\kappa+2}} \quad (\sigma>\kappa)$$

$$= \sum_{\substack{m=-\infty\\ (m,n)=1}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\chi(n)}{|\,mN\tau+n\,|^{2s-2\kappa+2}}$$

$$= \frac{1}{L(2s-2\kappa+2,\chi)}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\chi(n)}{|\,mN\tau+n\,|^{2s-2\kappa+2}}$$

$$= \frac{1}{L(2s-2\kappa+2,\chi)}\sum_{d\,|\,N_2}\mu(d)\,X(d)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\,\frac{X(n)}{|\,mN\tau+nd\,|^{2s-2\kappa+2}}$$

$$(5\cdot1\cdot10)$$

$$= \sum_{d\,|\,N_2}\frac{\mu(d)\,X(d)}{d^{2s-2\kappa+2}}\,\frac{T(s,\chi,d)}{L(2s-2\kappa+2,\chi)},$$

say, where $\mu(d)$ is Möbius's function. The dash denotes that the term with $m = n = 0$ is to be excluded.

5·2. Define

$$j(s,\psi,q) = \sum_{d\,|\,q}L(2s-2\kappa+2,\psi\chi_d)\sum_{\substack{m=1\\ (md,n,N)=1}}^{N/d}\sum_{n=1}^{N}\left(\frac{d}{q}\right)^{2s-2\kappa+2}\psi(m)f_{md,n}(s), \qquad (5\cdot2\cdot1)$$

where ψ is any character modulo k, kq divides N, and χ_d is the principal character modulo d. Then, by (5·1·7) and (5·1·8), we have

$$N^{s+1}(4\pi)^{-s}\Gamma(s)\,j(s,\psi,q) = \rho\iint_D y^{s-1}J(s,\tau,\psi,q)\,|\,H(\tau)\,|^2\,dx\,dy, \quad (5\cdot2\cdot2)$$

for $\sigma>\kappa$, where

$$J(s,\tau,\psi,q) = \sum_{d\,|\,q}\left(\frac{d}{q}\right)^{2s-2\kappa+2}L(2s-2\kappa+2,\psi\chi_d)\sum_{\substack{m=-\infty\\ (md,n)=1}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\psi(m)}{|\,md\tau+n\,|^{2s-2\kappa+2}}$$

$$= \sum_{d\,|\,q}\left(\frac{d}{q}\right)^{2s-2\kappa+2}\sum_{\substack{m=-\infty\\ (n,d)=1}}^{\infty}\sum_{n=-\infty}^{\infty}{}'\,\frac{\psi(m)}{|\,md\tau+n\,|^{2s-2\kappa+2}}$$

$$= \sum_{d\,|\,q}\sum_{d_1\,|\,d}\mu(d_1)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\,\frac{\psi(m)}{\left|\,mq\tau+nd_1\dfrac{q}{d}\right|^{2s-2\kappa+2}}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\,\frac{\psi(m)}{|\,mq\tau+n\,|^{2s-2\kappa+2}}, \qquad (5\cdot2\cdot3)$$

by the Möbius inversion formula.

By (5·1·10) and Lemma 4, we have

$$\left(\frac{N_1 N y}{\pi d}\right)^{s-\kappa+1} \Gamma(s-\kappa+1)\, T(s,\chi,d)$$
$$= \int_0^\infty w^{s-\kappa} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X(n) \exp\left\{-\frac{\pi w d}{N_1 N y}\left|m\frac{N}{d}\tau+n\right|^2\right\} dw$$
$$= \int_1^\infty w^{s-\kappa} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X(n) \exp\left\{-\frac{\pi w d}{N_1 N y}\left|m\frac{N}{d}\tau+n\right|^2\right\} dw$$
$$+\epsilon(\chi) N_1^{\frac{1}{2}} \int_1^\infty w^{\kappa-s-1} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{X}(m) \exp\left\{-\frac{\pi w d}{N_1 N y}\left|m\frac{N}{d}\tau+nN_1\right|^2\right\} dw \tag{5·2·4}$$

for $\sigma > \kappa$, when $\chi \neq \chi_0$. By Lemma 3, the right-hand side of (5·2·4) is an integral function of s, and hence, by (5·1·10), so is

$$\Gamma(s-\kappa+1)\, L(2s-2\kappa+2)\, G(s,\tau,\chi).$$

Therefore, by (5·1·9) and Lemmas 1 and 3, it follows that

$$\phi(s,\chi) = \left(\frac{2\pi}{N}\right)^{-2s} \Gamma(s)\,\Gamma(s-\kappa+1)\, L(2s-2\kappa+2,\chi)\, g(s,\chi)$$

is a regular function of s over the whole plane.

If χ is the principal character χ_0, we have, putting $\bar{N} = N/d$,

$$T(s,\chi_0,d) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{|m\bar{N}\tau+n|^{2s-2\kappa+2}},$$

where the dash denotes that the term with $m = n = 0$ is to be omitted, and we obtain

$$\left(\frac{Ny}{\pi d}\right)^{s-\kappa+1} \Gamma(s-\kappa+1)\, T(s,\chi_0,d) = \int_0^\infty w^{s-\kappa} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \exp\left\{-\frac{\pi w d}{N y}|m\bar{N}\tau+n|^2\right\} dw$$
$$= \int_1^\infty (w^{s-\kappa}+w^{\kappa-s-1}) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \exp\left\{-\frac{\pi w d}{N y}|m\bar{N}\tau+n|^2\right\} dw + \frac{1}{(s-\kappa)(s-\kappa+1)}. \tag{5·2·5}$$

By Lemma 3, the first term on the right-hand side of (5·2·5) is an integral function of s, and hence, as before, by (5·1·9), (5·1·10), (5·2·5), Lemmas 1 and 3, $\phi(s,\chi_0)$ is a regular function of s over the whole plane except for simple poles at the points $s = \kappa - 1$, and $s = \kappa$. The residue of $g(s,\chi_0)$ at $s = \kappa$ is

$$\pi\rho \frac{N^{-\kappa-1}(4\pi)^\kappa}{\Gamma(\kappa)\, L(2,\chi_0)} \sum_{d|N} \frac{\mu(d)}{dN} \iint_D y^{\kappa-2} |H(\tau)|^2\, dx\, dy$$
$$= 24\rho \frac{(4\pi)^{\kappa-1}}{N^{\kappa+3}\Gamma(\kappa)} \phi(N) \prod_{p|N}\left(1-\frac{1}{p^2}\right)^{-1} \iint_D y^{\kappa-2}|H(\tau)|^2\, dx\, dy$$
$$= \kappa\alpha\phi(N), \tag{5·2·6}$$

say. The value of α does not depend on the particular fundamental region integrated over, and remains the same if $H_V(\tau)$ is substituted for $H(\tau)$ in the integrand. Also it agrees with (1·4) in virtue of Remark C. It follows from (5·2·6) and (5·1·4) that $f(s)$ has a pole of residue $\kappa\alpha$ at $s=\kappa$, and, similarly, so has $f_{\gamma,\delta}(s)$ for any γ, δ. Part (vii) of Theorem 4 follows from (5·1·9), (5·1·10), (5·2·4) and (5·2·5). This completes the proof of the theorem with the exception of part (iv).

By (5·2·5),

$$\left(\frac{Ny}{\pi d}\right)^{s-\kappa+1}\Gamma(s-\kappa+1)\,T(s,\chi_0,d)=\left(\frac{Ny}{\pi d}\right)^{\kappa-s}\Gamma(\kappa-s)\,T(2\kappa-1-s,\chi_0,d). \tag{5·2·7}$$

If $\sigma<\kappa-1$, we have, by (5·2·4) and (5·2·7),

$$\begin{aligned}&\left(\frac{NN_1y}{\pi d}\right)^{s-\kappa+1}\Gamma(s-\kappa+1)\,T(s,\chi,d)\\
&\quad=\epsilon(X)\,N_1^{\frac{1}{2}}\left(\frac{NN_1y}{\pi d}\right)^{\kappa-s}\Gamma(\kappa-s)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\frac{\bar{X}(m)}{|m\bar{N}\tau+nN_1|^{2\kappa-2s}}\\
&\quad=\epsilon(X)\,N_1^{\frac{1}{2}}\left(\frac{Ny}{\pi N_1 d}\right)^{\kappa-s}\Gamma(\kappa-s)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}'\frac{\bar{X}(m)}{|m\bar{N}_2\tau+n|^{2\kappa-2s}}\\
&\quad=\epsilon(X)\,N_1^{\frac{1}{2}}\left(\frac{Ny}{\pi N_1 d}\right)^{\kappa-s}\Gamma(\kappa-s)\,J(2\kappa-1-s,\tau,\bar{X},\bar{N}_2),\end{aligned}$$

by (5·2·3), where $\bar{N}_2=N_2/d$. Therefore

$$\Gamma(s-\kappa+1)\,T(s,\chi,d)=\frac{\epsilon(X)}{\sqrt{N_1}}\left(\frac{Ny}{\pi d}\right)^{2\kappa-2s-1}\Gamma(\kappa-s)\,J(2\kappa-1-s,\tau,\bar{X},\bar{N}_2).$$

Hence, by (5·1·10),

$$\begin{aligned}&G(s,\tau,\chi)\,L(2s-2\kappa+2,\chi)\,\Gamma(s-\kappa+1)\\
&\qquad=\frac{\epsilon(X)}{\sqrt{N_1}}\sum_{d\mid N_2}\frac{\mu(d)\,X(d)}{d}\left(\frac{Ny}{\pi}\right)^{2\kappa-2s-1}\Gamma(\kappa-s)\,J(2\kappa-1-s,\tau,\bar{X},\bar{N}_2).\end{aligned}$$

And, by (5·1·9) and (5·2·2),

$$\begin{aligned}&N^{s+1}(4\pi)^{-s}\,\Gamma(s)\,\Gamma(s-\kappa+1)\,L(2s-2\kappa+2,\chi)\,g(s,\chi)\\
&\quad=\rho\frac{\epsilon(X)}{\sqrt{N_1}}\Gamma(\kappa-s)\sum_{d\mid N_2}\frac{\mu(d)\,X(d)}{d}\left(\frac{N}{\pi}\right)^{2\kappa-2s-1}\\
&\qquad\times\iint_D y^{2\kappa-s-2}\,|\,H(\tau)\,|^2\,J(2\kappa-1-s,\tau,\bar{X},\bar{N}_2)\,dx\,dy\\
&\quad=\frac{\epsilon(X)}{\sqrt{N_1}}\left(\frac{N}{\pi}\right)^{2\kappa-2s-1}\Gamma(\kappa-s)\,\Gamma(2\kappa-1-s)\\
&\qquad\times\sum_{d\mid N_2}\frac{\mu(d)\,X(d)}{d}N^{2\kappa-s}(4\pi)^{s-2\kappa+1}j(2\kappa-1-s,\bar{X},\bar{N}_2).\end{aligned}$$

Therefore, by (5·2·1),

$$\left(\frac{2\pi}{N}\right)^{-2s} \Gamma(s)\,\Gamma(s-\kappa+1)\,L(2s-2\kappa+2,\chi)\,g(s,\chi)$$

$$= \frac{\epsilon(X)}{\sqrt{N_1}}\left(\frac{2\pi}{N}\right)^{-2(2\kappa-1-s)} \Gamma(2\kappa-1-s)\,\Gamma(\kappa-s) \sum_{d\mid N_2} \frac{\mu(d)\,X(d)}{d}\, j(2\kappa-1-s,\bar{X},\bar{N}_2)$$

$$= \frac{\epsilon(X)}{\sqrt{N_1}}\left(\frac{2\pi}{N}\right)^{-2(2\kappa-1-s)} \Gamma(2\kappa-1-s)\,\Gamma(\kappa-s) \sum_{d\mid N_2} \frac{\mu(d)\,X(d)}{d} \sum_{dd_1\mid N_2} L(2\kappa-2s,\bar{X}\chi_{d_1})$$

$$\times\left(\frac{dd_1}{N_2}\right)^{2\kappa-2s} \sum_{\substack{m=1 \\ (md_1,\,n,\,N)=1}}^{N/d_1} \sum_{n=1}^{N} \bar{X}(m)\, f_{md_1,\,n}(2\kappa-1-s),$$

and (5·1·5) follows from this, since

$$\sum_{dd_1\mid N_2} \frac{\mu(d)\,X(d)}{d}\, d^{2\kappa-2s} = \prod_{p\mid N_2/d_1}\left(1-\frac{X(p)}{p^{2s-2\kappa+1}}\right).$$

5·3. *Proof of Theorems* 1 *and* 2 *for* $N>1$. Let

$$L(2s-2\kappa+2,\chi)\,g(s,\chi) = \sum_{n=1}^{\infty} \frac{b(n,\chi)}{n^s} \quad (\sigma>\kappa),$$

$$g(s,\chi) = \sum_{n=1}^{\infty} \frac{d(n,\chi)}{n^s}.$$

Then
$$|a_n|^2 = \frac{1}{\phi(N)} \sum_{\chi} d(n,\chi), \tag{5·3·1}$$

and
$$d(n,\chi) = \sum_{d^2\mid n} b\left(\frac{n}{d^2},\chi\right) \mu(d)\,\chi(d)\,d^{2\kappa-2}. \tag{5·3·2}$$

Put
$$\zeta(2-2s) \sum_{d\mid N_2} d^{2-2s} \prod_{p\mid N_2/d}\left(1+\frac{1}{p^{2s-1}}\right) \sum_{\substack{m=1 \\ (md,\,n,\,N)=1}}^{N/d} \sum_{n=1}^{N} f_{md,\,n}(\kappa-s) = \sum_{n=1}^{\infty} \frac{h_n n^s}{N_2^{2s}}$$

for $\sigma<0$. Then, as in § 4·2, we have

$$\sum_{n\leqslant x} |a_{\gamma,\delta}(n)|^2 = O(x^\kappa),$$

and
$$\sum_{n\leqslant x} n h_n = O(x). \tag{5·3·3}$$

We now apply Landau's theorem, as before, taking

$$Z(s) = g(s+\kappa-1,\chi)\,L(2s,\chi),$$

$$Z_0(s) = g(s+\kappa-1,\chi_0)\,L(2s,\chi_0).$$

Then
$$\beta = 1, \quad c_n = b(n,\chi)\,n^{1-\kappa}, \quad d_n = b(n,\chi_0)\,n^{1-\kappa}.$$

Also, by (5·1·5), we have

$$\Gamma(s)\,\Gamma(s+\kappa-1)\,Z(s) = \Gamma(1-s)\,\Gamma(\kappa-s)\sum_{n=1}^{\infty} e_n\lambda_n^s$$

for $\sigma < 0$, and, by (5·3·3),

$$\sum_{\lambda_n\leqslant x} |\,e_n\,|\,\lambda_n = O\big(\sum_{n\leqslant x} nh_n\big) = O(x).$$

Then, by Theorem 4, $Z(s)$ and $Z_0(s)$ satisfy conditions I, II, ..., VII, VIII′ and IX of Landau's theorem, with the same values of η, A, P, g and κ' as in § 4·2. And

$$R(x) = \kappa\alpha x\phi(N)\,L(2,\chi_0)\,E(\chi) = \gamma\kappa x,$$

say, by (5·2·6), where $E(\chi_0) = 1$, and $E(\chi) = 0$ when $\chi \neq \chi_0$. Hence

$$\sum_{n\leqslant x} c_n = \sum_{n\leqslant x} b(n,\chi)\,n^{1-\kappa} = \gamma\kappa x + O(x^{\frac{3}{5}}).$$

From this it follows that*

$$\sum_{n\leqslant x} b(n,\chi) = \gamma x^{\kappa} + O(x^{\kappa-\frac{3}{5}}).$$

Finally, by (5·3·2),

$$\begin{aligned}\sum_{n\leqslant x} d(n,\chi) &= \sum_{n\leqslant x}\sum_{d^2\mid n} b\left(\frac{n}{d^2},\chi\right)\mu(d)\,\chi(d)\,d^{2\kappa-2}\\ &= \sum_{d\leqslant\sqrt{x}} \mu(d)\,\chi(d)\,d^{2\kappa-2}\sum_{\lambda\leqslant x/d^2} b(\lambda,\chi)\\ &= \sum_{d\leqslant\sqrt{x}} \mu(d)\,\chi(d)\,\{\gamma x^{\kappa}d^{-2} + O(x^{\kappa-\frac{3}{5}}d^{-\frac{4}{5}})\}\\ &= \frac{\gamma x^{\kappa}}{L(2,\chi)} + O(x^{\kappa-\frac{1}{2}}) + O(x^{\kappa-\frac{1}{2}})\\ &= \alpha\phi(N)\,E(\chi)\,x^{\kappa} + O(x^{\kappa-\frac{1}{2}}).\end{aligned}$$

And therefore, by (5·3·1), Theorem 1 follows.

* This is trivial when $\kappa > \frac{3}{5}$. It is true also for $\kappa \leqslant \frac{3}{5}$, since Landau's theorem can be extended to show that

$$\sum_{n\leqslant x} c_n n^a = R(a,x) + O(x^{\kappa'+a}\log^g x),$$

for any real $a > -\beta$, where $R(a,x)$ is the sum of the residues of $x^s Z(s-a)/s$ in the strip $\kappa' \leqslant \sigma - a \leqslant \beta$.

I write κ' where Landau has κ to avoid confusion with the dimension $-\kappa$.

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